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Implicitization Matrices in the Style of Sylvester with the Order of Bézout

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Abstract. Resultants are the standard tool used to compute the implicit equation of a rational curve or surface. Here we present a new way to compute the implicit equation of a rational curve by taking the determinant of a matrix having the style of the Sylvester resultant but the size of the Bézout resultant. Thus the new method has the advantages of both resultant schemes, representing the implicit equation as the determinant of a matrix with simple linear entries and lots of zeros just like the Sylvester resultant, but with the same small size as the Bézout resultant.

§1. Implicitization and Resultants

In Computer Aided Geometric Design (CAGD), curves and surfaces have two standard representations: parametric and implicit. The parametric representation is convenient for rendering curves and surfaces, whereas the implicit representation is useful for checking whether or not a point lies on a curve or surface. In the ideal situation, both representations are available. Given the parametric form of a curve or surface, one basic problem in CAGD is implicitization — that is, to find the implicit representation. Resultants are an effective tool for solving this problem for rational curves and surfaces [4,5].

Resultants are polynomial expressions in the coefficients of a set of polynomials; the vanishing of these expressions signals that the set of polynomials have a common root. For two univariate polynomials, there are two standard resultant formulations: Sylvester's resultant and Bézout's resultant. Given two degree n polynomials

$$f = \sum_{i=0}^{n} a_i t^i, \qquad g = \sum_{i=0}^{n} b_i t^i,$$

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the Sylvester resultant is the determinant of the $2n \times 2n$ matrix

$$Syl(f,g) = \begin{bmatrix} a_0 & b_0 & & & & & \\ a_1 & b_1 & a_0 & b_0 & & & \\ \vdots & \vdots & a_1 & b_1 & \ddots & & \\ a_{n-1} & b_{n-1} & \vdots & \vdots & \ddots & a_0 & b_0 \\ a_n & b_n & a_{n-1} & b_{n-1} & \ddots & a_1 & b_1 \\ & & a_n & b_n & \ddots & \vdots & \vdots \\ & & & & \ddots & a_{n-1} & b_{n-1} \\ & & & & & a_n & b_n \end{bmatrix}.$$

Thus, the Sylvester resultant is just the determinant of the coefficient matrix of the polynomials $f, g, \cdots, t^{n-1}f, t^{n-1}g$ [6,9]. The Bézout resultant of f and g is the determinant of the $n \times n$ coefficient matrix $Bez(f,g)$, where $Bez(f,g)$ is defined by

$$\frac{\begin{vmatrix} f(t) & g(t) \\ f(\alpha) & g(\alpha) \end{vmatrix}}{\alpha - t} = [1 \quad \cdots \quad t^{n-1}] \cdot Bez(f,g) \cdot [1 \quad \cdots \quad \alpha^{n-1}]^T.$$

Explicit entry formulas for the Bézout resultant and fast computational algorithms for these entries can be found in [4,1].

The Sylvester and Bézout resultants each have certain advantages and disadvantages. The Sylvester resultant is sparse and all the nonzero entries of the Sylvester resultant come directly from the coefficients of f or g. The entries of the Bézout resultant are more complicated. However, to calculate the Sylvester resultant, a large determinant has to be computed, whereas the Bézout resultant matrix is much more compact.

To see why resultants arise naturally in implicitization, consider a rational curve

$$X = \frac{x(t)}{w(t)}, \qquad Y = \frac{y(t)}{w(t)}, \tag{1}$$

where $x(t), y(t), w(t)$ are polynomials. To obtain the implicit representation $F(X,Y) = 0$ for curve (1), introduce two auxiliary polynomials (in t)

$$X \cdot w(t) - x(t), \qquad Y \cdot w(t) - y(t). \tag{2}$$

By definition, the resultant of these two polynomials vanishes if and only if they have a common root, i.e. if and only if the point (X,Y) satisfies the two equations

$$X \cdot w(t) - x(t) = 0, \qquad Y \cdot w(t) - y(t) = 0,$$

for some value of t. Thus, (X,Y) makes the resultant of $X \cdot w(t) - x(t)$, $Y \cdot w(t) - y(t)$ vanish if and only if (X,Y) is on curve (1). So setting the resultant to zero yields the implicit equation of the parametric curve.

But which form of the resultant should one use? The Sylvester resultant has simple linear entries and lots of zeros, but to calculate the Sylvester resultant a large determinant has to be computed. The Bézout resultant has a more compact form, but the entries are much more complicated than the entries of the Sylvester resultant. Here we present a new way to compute the implicit equation of a rational curve by taking the determinant of a matrix having the style of the Sylvester resultant but the size of the Bézout resultant. Thus the new method has the advantages of both resultant schemes. That is, the new approach represents the implicit equation as the determinant of a matrix with simple linear entries and lots of zeros just like the Sylvester resultant, but the matrix has the same small size as the Bézout resultant.

Surfaces are beyond the scope of this work, but we hope to develop similar techniques for rational surfaces in a future paper [2].

§2. Implicitization from Moving Lines

In this section, we consider first rational curves of even degree. We begin by reviewing the concept of a moving line that follows a rational curve [7,8], and we show that there are always at least two moving lines of degree m that follow a rational curve of degree $2m$. The $m \times m$ Bézout determinant of these two moving lines has been used by previous authors to establish the efficacy of implicitization by the method of moving conics [3,8]. Here we prove that the $2m \times 2m$ Sylvester determinant of these two moving lines is an implicit expression for the rational curve if and only if there are no moving lines of degree $< m$ that follow the curve. This construction generates an implicitization matrix in the style of Sylvester with the order of Bézout. At the end of this section, we develop similar results for rational curves of odd degree.

2.1 Even degree rational curves

A rational curve of degree $2m$ can be written as $(x(t) : y(t) : w(t))$, where

$$x(t) = \sum_{i=0}^{2m} a_i t^i, \qquad y(t) = \sum_{i=0}^{2m} b_i t^i, \qquad w(t) = \sum_{i=0}^{2m} c_i t^i \tag{3}$$

and $\gcd(x(t), y(t), w(t)) = 1$. We can think of a rational curve as the track of a moving point.

Analogously, a moving line of degree d is defined by an implicit equation of the form

$$(A_0 x + B_0 y + C_0 w) + \cdots + (A_d x + B_d y + C_d w)t^d = 0, \tag{4}$$

where the coefficients $A_0, B_0, C_0, \cdots, A_d, B_d, C_d$ are constants. We say that the moving line (4) follows the rational curve (3) if and only if

$$(A_0 x(t) + B_0 y(t) + C_0 w(t)) + \cdots + (A_d x(t) + B_d y(t) + C_d w(t))t^d \equiv 0. \tag{5}$$

For example, the equations

$$x \cdot w(t) - w \cdot x(t) = 0, \quad y \cdot w(t) - w \cdot y(t) = 0,$$

or equivalently,

$$X \cdot w(t) - x(t) = 0, \quad Y \cdot w(t) - y(t) = 0,$$

are two moving lines of degree $2m$ that follow the rational curve (3). Thus the standard way to find the implicit equation of the rational curve (3) is to compute the resultant of these two moving lines of degree $2m$ that follow the curve. To simplify the determinant that represents the implicit equation, we are going to take the resultant of two moving lines of degree m that follow the curve.

By equating the coefficients of the powers of t in (5) to zero, we obtain $2m + d + 1$ equations in $3d + 3$ unknowns. The $3d + 3$ unknowns $A_0, B_0, C_0, \cdots, A_d, B_d, C_d$ of the moving line (4) can be found by solving the $(2m + d + 1) \times (3d + 3)$ linear system

$$\begin{aligned}\texttt{Coeff}(\, x(t), \quad y(t), \quad w(t), \quad \cdots, \quad x(t)t^d, \quad y(t)t^d, \quad w(t)t^d \,)& \\ \cdot \, [\, A_0 \quad B_0 \quad B_0 \quad \cdots \quad A_d \quad B_d \quad C_d \,] = 0&,\end{aligned}$$

where "**Coeff**" stands for the matrix whose columns are the coefficients of the given polynomials. When $d = m$, the dimension of the linear system is $(3m+1) \times (3m+3)$. Consequently, there are at least two linearly independent solutions $p(x, y, w; t)$ and $q(x, y, w; t)$.

The $2m \times 2m$ Sylvester matrix $Syl(p, q)$ obtained by eliminating t from p and q can be written as

$$Syl(p,q) = \texttt{Coeff}\,(\, p, \quad q, \quad p\,t, \quad q\,t, \quad \cdots, \quad p\,t^{m-1}, \quad q\,t^{m-1}\,).$$

Theorem 1. *$|Syl(p,q)| = 0$ is the implicit equation of the rational curve (3) when there are no moving lines of degree $< m$ that follow curve (3).*

Proof: Since the implicit equation of a rational curve of degree $2m$ is represented by an irreducible polynomial of degree $2m$ [8], we need only establish three facts:

1) $|Syl(p,q)| \not\equiv 0$,
2) $|Syl(p,q)|$ is of degree at most $2m$,
3) $|Syl(p,q)|$ vanishes on $(x(t) : y(t) : w(t))$.

From the properties of resultants, we know that $|Syl(p,q)| \equiv 0$ if and only if p and q have a common factor $g(t)$ of degree ≥ 1. Since p and q are of degree 1 in x, y, w, one of g and p/g would be of degree 1 in x, y, w, i.e. a moving line of degree $< m$ that follows the curve. But by assumption there are no such moving lines, so $|Syl(p,q)|$ cannot vanish identically.

Since $|Syl(p,q)|$ is the determinant of a $2m \times 2m$ matrix with linear entries in x, y, w, obviously the degree of $|Syl(p,q)|$ is at most $2m$ in x, y, w. Finally, $p(x,y,w;t)$ and $q(x,y,w;t)$ follow the rational curve, so

$$p(x(t_0), y(t_0), w(t_0); t_0) \equiv 0, \qquad q(x(t_0), y(t_0), w(t_0); t_0) \equiv 0,$$

for any parameter t_0. That is, the two polynomials

$$p(x(t_0), y(t_0), w(t_0); t), \qquad q(x(t_0), y(t_0), w(t_0); t)$$

have a common root t_0. Hence, the resultant

$$|Syl(p(x(t_0), y(t_0), w(t_0); t),\ q(x(t_0), y(t_0), w(t_0); t))| = 0.$$

Therefore, $|Syl(p,q)|$ vanishes on $(x(t) : y(t) : w(t))$. □

In summary, we have shown that for a degree $2m$ rational curve, the $2m \times 2m$ Sylvester determinant of two degree m moving lines is the implicit equation of the curve if there are no moving lines of degree $< m$ that follow the curve. The existence of a moving line of degree $m-1$ that follows the curve is equivalent to the vanishing of the $3m \times 3m$ determinant

$$|\texttt{Coeff}(\,x(t),\quad y(t),\quad w(t),\quad \cdots,\quad t^{m-1}x(t),\quad t^{m-1}y(t),\quad t^{m-1}w(t)\,)|\,.$$

This determinant is a polynomial in the coefficients of $x(t), y(t), w(t)$ and therefore almost never vanishes. However, in case such lower degree moving lines do exist, the desired Sylvester determinant can be salvaged by finding the μ−basis (see Section 3).

2.2 An example

Consider the rational sextic curve

$$x(t) = 1 + 2t^2 + 2t^5, \quad y(t) = 2 + t^6, \quad w(t) = 1 + t + 2t^2 + 2t^3 + t^4 + t^6.$$

To use the standard method to implicitize this curve, we introduce two auxiliary polynomials

$$X \cdot w(t) - x(t), \quad Y \cdot w(t) - y(t).$$

Their Sylvester resultant is the 12×12 determinant

$$\begin{vmatrix}
-1+X & -2+Y & 0 & 0 & \cdots & 0 & 0 \\
X & Y & -1+X & -2+Y & \cdots & 0 & 0 \\
-2+2X & 2Y & X & Y & \cdots & 0 & 0 \\
2X & 2Y & -2+2X & 2Y & \cdots & 0 & 0 \\
X & Y & 2X & 2Y & \cdots & 0 & 0 \\
-2 & 0 & X & Y & \cdots & -1+X & -2+Y \\
X & -1+Y & -2 & 0 & \cdots & X & Y \\
0 & 0 & X & -1+Y & \cdots & -2+2X & 2Y \\
0 & 0 & 0 & 0 & \cdots & 2X & 2Y \\
0 & 0 & 0 & 0 & \cdots & X & Y \\
0 & 0 & 0 & 0 & \cdots & -2 & 0 \\
0 & 0 & 0 & 0 & \cdots & X & -1+Y
\end{vmatrix},$$

where the six columns in the middle have been omitted. The Bézout resultant is the 6×6 determinant [4,1]

$$\begin{vmatrix} 2X-Y & -4+4X & 4X-2Y & 2X-Y & -4+2Y & 1+X-Y \\ -4+4X & 4X & 2X-Y & -4+2Y & 1+X+Y & -X \\ 4X-2Y & 2X-Y & -4-2Y & 1+X-Y & -X+4Y & 2-2X-2Y \\ 2X-Y & -4+2Y & 1+X-Y & -X+4Y & 2-2X+2Y & -2X \\ -4+2Y & 1+X+Y & -X+4Y & 2-2X+2Y & -2X+2Y & -X \\ 1+X-Y & -X & 2-2X-2Y & -2X & -X & 2-2Y \end{vmatrix}.$$

On the other hand, using linear algebra, it is easy to calculate two moving lines of degree three following this curve:

$$\begin{aligned} &(855w + 31x - 443y) + t\,(77y - 778w - 231x) \\ &\qquad + t^2(338x - 666y) + t^3(25w + 333x - 25y) = 0, \\ &(780w - 413y + 46x) + t\,(-748w + 82y - 196x) \\ &\qquad + t^2(25w - 631y + 333x) + t^3(303x) = 0. \end{aligned}$$

The new method computes the implicit equation for this curve by taking the 6×6 Sylvester determinant of these two moving lines:

$$\begin{vmatrix} -443y + 31x + 855w & 46x - 413y + 780w & 0 & 0 & 0 & 0 \\ -778w - 231x + 77y & -748w + 82y - 196x & -443y + 31x + 855w & 46x - 413y + 780w & 0 & 0 \\ 338x - 666y & 25w - 631y + 333x & -778w - 231x + 77y & -748w + 82y - 196x & -443y + 31x + 855w & 46x - 413y + 780w \\ 25w + 333x - 25y & 303x & 338x - 666y & 25w - 631y + 333x & -778w - 231x + 77y & -748w + 82y - 196x \\ 0 & 0 & 25w + 333x - 25y & 303x & 338x - 666y & 25w - 631y + 333x \\ 0 & 0 & 0 & 0 & 25w + 333x - 25y & 303x \end{vmatrix}.$$

Using Mathematica, we verified that all three methods produce the correct implicit equation for the given rational curve. Notice that the determinant generated by the new method has the structure of the Sylvester resultant but the order of the Bézout resultant.

2.3 Odd degree rational curves

For a rational curve of degree $2m + 1$, there is always at least one non-zero moving line of degree m and at least 3 linearly independent moving lines of degree $m + 1$ that follow the curve. Therefore, there always exists a moving line p of degree m and a moving line q of degree $m + 1$, where q is not a multiple of p, that follow the rational curve. Suppose there is no moving line of degree $< m$ that follows the curve. Then by an argument similar to the case of even degrees, the Sylvester resultant of p and q is the determinant of a $(2m + 1) \times (2m + 1)$ matrix that represents the implicit equation of the rational curve.

§3. Anti-Annihilation by μ–Basis

The implicitization method in Section 2 works when there are no low degree moving lines that follow the curve. In the rare cases when there do exist low degree moving lines following the curve, the Sylvester resultant used in Section 2 generally vanishes identically [7]. In order to circumvent this difficulty— that is, to counter the annihilation effect of low degree moving lines— and show how the desired Sylvester-style/Bézout-size determinant can still be obtained, we need the notion of a μ–basis [3].

Consider a degree n rational curve $(x(t) : y(t) : w(t))$. By solving an $(n+d+1) \times (3d+3)$ linear system [Section 2.1], we find that the number of linearly independent degree d moving lines that follow this curve is at least $(3d+3)-(n+d+1) = 2d+2-n$. Thus the system always has solutions when $3d+3 > n+d+1$ or $d > n/2-1$. Hence if μ is the lowest degree in t of all the moving lines that follow the curve, then $\mu \leq \lfloor n/2 \rfloor$. Let p be a moving line with the lowest degree μ that follows the curve.

By our previous analysis, there are at least $2(n-\mu)+2-n = n+2-2\mu$ linearly independent moving lines of degree $n-\mu$ that follow the curve. Not all of them can be multiples of p because p can only generate $n+1-2\mu$ independent moving lines of degree at most $n-\mu$: $p, \cdots, pt^{n-2\mu}$. Hence there is a degree $n-\mu$ moving line q that is not a multiple of p.

The two moving lines p and q that we just constructed have the following nice property:

Theorem 2. *Any degree d moving line l that follows the curve $(x(t) : y(t) : w(t))$ can be written uniquely as $Ap + Bq$, where A is a polynomial in t of degree at most $d-\mu$, and B is a polynomial in t of degree at most $d+\mu-n$ [3].*

Proof: [3] presents a proof of this result based on ideal theory. Here we provide a simpler proof using only linear algebra. A degree d moving line can always be written as

$$l = l_x(t)x + l_y(t)y + l_w(t)w,$$

where l_x, l_y, l_w are polynomials in t of degree at most d. It will be very convenient in the following discussion to treat a moving line l as a vector $\vec{l} = (l_x(t), l_y(t), l_w(t))$. Furthermore, note that since the components of the vector $\vec{l}$ are polynomials, the scalar field is the field of rational functions in t.

Let

$$\vec{r} = (x(t), y(t), w(t)), \quad \vec{p} = (p_x(t), p_y(t), p_w(t)), \quad \vec{q} = (q_x(t), q_y(t), q_w(t)).$$

Since, by assumption, the dot products $\vec{p} \cdot \vec{r} = \vec{q} \cdot \vec{r} = 0$, the vector $\vec{r}$ is proportional to the cross product $\vec{p} \times \vec{q}$. That is,

$$\vec{r} = \frac{u(t)}{v(t)}\, \vec{p} \times \vec{q} \tag{6}$$

where $\gcd(u, v) = 1$. Since at least one component of $\vec{r}$ is of degree n and all the components of $\vec{p} \times \vec{q}$ are of degree at most n, the degree of u is at least as great as the degree of v. Moreover, by (6), $u(t)$ divides each component of $\vec{r}$; thus $u(t)$ divides $\gcd(x(t), y(t), w(t)) = 1$. Therefore, the degree of u and the degree of v are both zero, so

$$\vec{r} = \lambda\, \vec{p} \times \vec{q}, \tag{7}$$

where λ is a constant.

Let l be a degree d moving line following the curve. We have $\vec{l} \cdot \vec{r} = 0$ and $\vec{p} \cdot \vec{r} = 0$, $\vec{q} \cdot \vec{r} = 0$, Thus $\vec{l}$, $\vec{p}$, $\vec{q}$ are linearly dependent. Since p and q are linearly independent, we can write

$$\vec{l} = A(t)\vec{p} + B(t)\vec{q}, \tag{8}$$

where $A(t)$ and $B(t)$ are rational functions in t. By (8) and (7), we have

$$\vec{l} \times \vec{q} = A(t)\vec{p} \times \vec{q} = \frac{A(t)}{\lambda}\vec{r}. \tag{9}$$

If $A(t)$ is a polynomial, its degree is at most $d - \mu$ because all the components of $\vec{l} \times \vec{q}$ are of degree at most $d + n - \mu$ and at least one component of $\vec{r}$ is of degree n.

Next we show that $A(t)$ is indeed a polynomial. Since $\gcd(x(t), y(t), w(t))$ is equal to 1, there exist polynomials $x^*(t)$, $y^*(t)$, $w^*(t)$ such that

$$x(t)x^*(t) + y(t)y^*(t) + w(t)w^*(t) = 1.$$

Let $\vec{r}^* = (x^*(t), y^*(t), w^*(t))$. Applying $\vec{r} \cdot \vec{r}^* = 1$ to (9), we have

$$A(t) = \lambda \vec{l} \times \vec{q} \cdot \vec{r}^*. \tag{10}$$

Since the components of all the vectors on the right hand side of (10) are actually polynomials rather than rational functions in t, $A(t)$ must also be a polynomial in t.

The fact that $B(t)$ is a polynomial of degree at most $d + \mu - n$ can be established similarly.

Finally we show that $A(t)$ and $B(t)$ are unique for any given $\vec{l}$. Suppose we have

$$A_1(t)\vec{p} + B_1(t)\vec{q} = A_2(t)\vec{p} + B_2(t)\vec{q};$$

then $(A_1(t) - A_2(t))\vec{p} = (B_2(t) - B_1(t))\vec{q}$. If $B_2(t) - B_1(t)$ is not zero, then it divides $A_1(t) - A_2(t)$, otherwise $p = \vec{p} \cdot (x, y, w)$ is not a moving line with minimal degree that follows the curve. But this would mean that $\vec{q}$ is a multiple of $\vec{p}$; hence, the moving line q is a multiple of the moving line p, which is contrary to assumption. Therefore $A_1(t) = A_2(t)$ and $B_1(t) = B_2(t)$. □

The two moving lines p and q in Theorem 1 are called a μ−basis of the curve $(x(t):y(t):w(t))$.

We have shown in Section 2.1 that when there are no moving lines of degree $< m$ following a rational curve of degree $n = 2m$, there will be two moving lines of degree m following the curve and their Sylvester determinant gives the implicit equation. Clearly these two moving lines are simply p and q in Theorem 1 with $\mu = m = n - \mu$. Theorem 1 also tells us that for $\mu \le d < n - \mu$, a degree d moving line l has the form

$$l = c_0 p + c_1 pt + \cdots c_{d-\mu} pt^{d-\mu},$$

where c_i are constants because l is of the form $Ap + Bq$ with $B = 0$ due to the degree constraints on A and B. Consequently, the Sylvester determinant of any two of these degree d moving lines vanishes as both are multiples of p; furthermore, the number of such linearly independent degree d moving lines is $N_d = d - \mu + 1$. In particular, when there are moving lines of degree $< m$ that follow the curve, we have $\mu < m < n - \mu$, so the Sylvester resultant of any two degree m moving lines vanishes and there are $N_m = m - \mu + 1 \ge 2$ degree m moving lines following the curve. Note that we can find μ in terms of N_m:

$$\begin{aligned} \mu &= m - N_m + 1, \\ N_m &= 3m + 3 - \textit{Rank of} \\ &\qquad \mathtt{Coeff}(\, x(t),\, y(t),\, w(t),\, \cdots,\, t^m x(t),\, t^m y(t),\, t^m w(t)\,). \end{aligned}$$

In general then, for a degree n rational curve $(x(t):y(t):w(t))$, we can obtain the μ−basis functions p and q by straightforward linear algebra. Since p is irreducible (by degree minimality) and q is not a multiple of p, they have no common factors. Hence their Sylvester resultant

$$Syl(p,q) = \mathtt{Coeff}\,(p, \quad pt, \quad \cdots, \quad pt^{n-\mu-1}, \quad q, \quad qt, \quad \cdots, \quad qt^{\mu-1}\,)$$

is a matrix of size $n \times n$ whose determinant does not vanish identically. By the divisibility and degree argument of Section 2.1, we see that this Sylvester determinant gives an implicit expression for the rational curve $(x(t):y(t):w(t))$ in the style of Sylvester with the order Bézout.

As an example, consider the degree n rational curve

$$(x(t):y(t):w(t)) = (1 : t^{n-1} : 1 + t^n). \tag{11}$$

Simple calculations reveal that $p = x + ty - w$ and $q = t^{n-1}x - y$. The Sylvester determinant

$$|\,p \quad \cdots \quad pt^{n-2} \quad q\,| = \begin{vmatrix} x-w & & & -y \\ y & \ddots & & \vdots \\ & \ddots & x-w & \vdots \\ & & y & x \end{vmatrix} = (-1)^{n-1}y^n + x(x-w)^{n-1}$$

is easily seen to represent the implicit equation of this rational curve.

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